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Six hundred and seventeenth Meeting.

February 8, 1870. — MONTHLY MEETING.

The PRESIDENT in the chair.

The President called the attention of the Academy to the recent decease of Overbeck, at Rome, of the Foreign Honorary Members.

Dr. E. H. Clark made a communication on the results of an analysis of one thousand cases of disease in general practice, as to the curative action of drugs.

Dr. Bowditch made a communication, illustrated by a chart, on the apparent connection of cloudy days and mortality from consumption, for the period from 1811 to 1857, and in this vicinity.

Two papers by Mr. G. W. Hill were presented at the meeting of December 11, 1869.

The following problem seems to possess some interest, and I have not, in my reading, met with any discussion of it: —

To determine the elements of the orbit of a planet or satellite, which moves in a circle in the plane of the ecliptic, from three observations of its direction from the earth, made at equal intervals of time; the positions of the earth and the central body at these times being known, but the sum of the masses of the central body and the planet or satellite being unknown.

Or, geometrically stated, —

In a plane, given a point as centre and three straight lines, required to describe a circle, so that the arcs intercepted between the first and second, and the second and third, lines may be equal.

Let generally R denote the distance of the central body from the earth;
" " L its longitude as seen from the earth;
" " r the radius of the orbit of the planet;
" " λ its longitude as seen from the earth;
" " χ its longitude as seen from the central body.

Moreover, employ the subscripts $(_{-1})$, $(_0)$, $(_1)$, to denote the special values of the above quantities, which have place respectively at the three times of observation in their order.

If a perpendicular be let fall from the central body on the straight line which joins the earth and the body whose orbit is to be determined, its length is obviously

$$R \sin (\lambda - L) ;$$

another expression for the length of the same line is

$$r \sin (\chi - \lambda).$$

Hence for the three times of observation, the three equations

$$\begin{aligned} r \sin (\chi_{-1} - \lambda_{-1}) &= R_{-1} \sin (\lambda_{-1} - L_{-1}), \\ r \sin (\chi_0 - \lambda_0) &= R_0 \sin (\lambda_0 - L_0), \\ r \sin (\chi_1 - \lambda_1) &= R_1 \sin (\lambda_1 - L_1). \end{aligned}$$

But since the orbit is circular, χ increases uniformly with the time, and consequently $\chi_0 - \chi_{-1} = \chi_1 - \chi_0 = \eta$ suppose.

Thus the above equations may be written

$$\begin{aligned} r \sin (\chi_0 - \eta - \lambda_{-1}) &= R_{-1} \sin (\lambda_{-1} - L_{-1}) = a_{-1}, \\ r \sin (\chi_0 - \lambda_0) &= R_0 \sin (\lambda_0 - L_0) = a_0, \\ r \sin (\chi_0 + \eta - \lambda_1) &= R_1 \sin (\lambda_1 - L_1) = a_1, \end{aligned}$$

which serve to determine the three unknown quantities r, χ_0, and η; and it will be noticed that their right-hand members are known quantities.

If the sum of the masses of the central body and the body whose orbit is sought is denoted by μ, and the common interval of time between the observations by t,

$$\eta = t \sqrt{\frac{\mu}{r^3}} ;$$

thus, if μ were known, two observations would suffice to determine the orbit; but if μ is not known, η must be regarded as an independent unknown quantity. Hence the necessity for the restriction put at the end of the statement of the problem. Also by this restriction the problem is made to depend on the solution of an algebraical equation instead of a transcendental one.

The equations can be simplified by taking two unknown quantities, ω and σ, instead of χ_0 and η, such that

$$\omega = \chi_0 - \frac{\lambda_1 + \lambda_{-1}}{2}$$

$$\sigma = \eta - \frac{\lambda_1 - \lambda_{-1}}{2}$$

and putting

$$\delta = \frac{\lambda_1 + \lambda_{-1}}{2} - \lambda_0.$$

Then the equations become

$$r \sin(\omega - \sigma) = a_{-1},$$
$$r \sin(\omega + \delta) = a_0,$$
$$r \sin(\omega + \sigma) = a_1.$$

Or,

$$r \sin\omega \cos\sigma = \frac{a_1 + a_{-1}}{2}.$$
$$r \sin(\omega + \delta) = a_0,$$
$$r \cos\omega \sin\sigma = \frac{a_1 - a_{-1}}{2}.$$

If $r \sin \omega$ and $r \cos \omega$ are eliminated from these equations, and we make

$$\frac{a_1 + a_{-1}}{2\,a_0} \cos\delta = \mathrm{a} = c \cos\beta,$$

$$\frac{a_1 - a_{-1}}{2\,a_0} \sin\delta = \mathrm{b} = c \sin\beta,$$

where c may be taken as positive and the quadrant of β becomes determinate, or β may be assumed between the limits ± 90°, there will be obtained, for the determination of σ, the equation

$$\sin 2\,\sigma = 2\,c \sin(\sigma + \beta).$$

The computation of c and β may be facilitated by introducing the auxiliary quantities k and ζ, such that

$$k \sin\zeta = \frac{a_{-1}}{\sqrt{2}\,a_0}$$

$$k \cos\zeta = \frac{a_1}{\sqrt{2}\,a_0},$$

then

$$c \cos\beta = k \cos(45^\circ - \zeta) \cos\delta.$$
$$c \sin\beta = k \sin(45^\circ - \zeta) \sin\delta.$$

It is evident that the determination of σ depends on the solution of an equation of the fourth degree; but its value can be very readily obtained from the above equation by the tentative process; and then r and ω by means of the equations

$$r \sin\omega = \frac{a_0\,k \cos(45^\circ - \zeta)}{\cos\sigma},$$

$$r \cos\omega = \frac{a_0\,k \sin(45^\circ - \zeta)}{\sin\sigma},$$

and finally χ_0 and η by means of the relations given above.

There is a very simple geometrical construction of the roots of the equation in σ. Making $\cos\sigma = \chi$, and $\sin\sigma = y$, the values of χ and y are the co-ordinates of the intersections of the curves whose equations are

$$\chi^2 + y^2 = 1,$$
$$(\chi - \mathrm{a})\,(y - \mathrm{b}) = \mathrm{a\,b}$$

Consequently, if we construct the equilateral hyperbola whose equation is

$$\chi y = \pm 1,$$

and from a point on it, whose co-ordinates are

$$\chi^1 = -\frac{\mathrm{a}}{\sqrt{\pm\,\mathrm{a\,b}}},$$

$$y^1 = -\frac{\mathrm{b}}{\sqrt{\pm\,\mathrm{a\,b}}},$$

as centre, we describe a circle, whose radius is $\frac{1}{\sqrt{\pm\,\mathrm{a\,b}}}$, and then draw radii to the points of intersection of the curves, the angles made by these radii with the χ axis of co-ordinates are the values of σ. Since the centre of the circle is on the hyperbola, there are at least two intersections, and thus the equation in σ has at least two real roots. The geometrical construction readily affords the condition which a and b must satisfy in order that there may be four real roots. The condition is, that the length of the straight line drawn from the point a, b, on the hyperbola whose equation is

$$\chi\, y = \mathrm{a\,b}$$

normal to the opposite branch, shall be less than unity. The equation to the normal which passes through the point χ'', y'' on this curve, is

$$\chi''\,(\chi - \chi'') - y''\,(y - y'') = 0.$$

The condition that it passes through the point a, b, gives

$$\chi''\,(\chi'' - \mathrm{a}) - y''\,(y'' - \mathrm{b}) = 0,$$
$$\chi''\, y'' = \mathrm{a\,b}.$$

If we multiply the first of these by χ''^2, we get

$$\chi''^3\,(\chi'' - \mathrm{a}) - \mathrm{a\,b}\,(\mathrm{a\,b} - \mathrm{b}\,\chi'') = 0,$$

or, rejecting the useless factor $\chi'' - \mathrm{a}$,

$$\chi''^3 + \mathrm{a}\,\mathrm{b}^2 = 0,$$

whence

$$\chi'' = -\sqrt[3]{\mathrm{a}\,\mathrm{b}^2},$$

and by interchanging a and b,

$$y'' = -\sqrt[3]{\mathrm{a}^2\,\mathrm{b}}.$$

And thus the length of the normal

$$\sqrt{(\chi'' - \mathrm{a})^2 + (y'' - \mathrm{b})^2} = \left[(\mathrm{a} + \sqrt[3]{\mathrm{a}\,\mathrm{b}^2})^2 + (\mathrm{b} + \sqrt[3]{\mathrm{a}^2\,\mathrm{b}})^2\right]^{\frac{1}{2}}$$
$$= [\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}}]^{\frac{3}{2}}.$$

Consequently,

if $\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} < 1$, there will be four real roots;
" $\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} = 1$, there will be four, and two will be equal;
" $\mathrm{a}^{\frac{2}{3}} + \mathrm{b}^{\frac{2}{3}} > 1$, there will be only two real roots.

We will now show how to arrive at a direct solution of the problem by the employment of trigonometric formulæ. If tan σ is taken for the unknown quantity, the equation, on which the solution of the problem depends, is

$$[c \cos\beta \tan\sigma + c \sin\beta]^2 (1 + \tan^2\sigma) = \tan^2\sigma,$$

or if we put $\tan\sigma = \chi$,

$$(\chi + \tan\beta)^2 (\chi^2 + 1) = \frac{\chi^2}{c^2 \cos^2\beta},$$

or, expanded,

$$\chi^4 + 2\tan\beta\,.\,\chi^3 + \frac{c^2 - 1}{c^2 \cos^2\beta}\,\chi^2 + 2\tan\beta\,.\,\chi + \tan^2\beta = 0.$$

A quantity μ may be assumed, such that this biquadratic shall be resolved into the two quadratics

$$\chi^2 + 2\,\frac{\sin\mu \cos(\beta + \mu)}{\cos\beta \cos 2\mu}\,\chi + \tan\beta \tan\mu = 0,$$

$$\chi^2 + 2\,\frac{\cos\mu \sin(\beta - \mu)}{\cos\beta \cos 2\mu}\,\chi + \tan\beta \cot\mu = 0.$$

That this is possible will be evident on multiplying the left-hand members of these equations together, for after some reductions easy to make, all the coefficients, with the exception of that of χ^2, will be found to be identical with those of the biquadratic; and consequently μ is determined by the equation

$$\tan\beta\,[\tan\mu + \cot\mu] + 2\,\frac{\sin 2\mu \sin(\beta - \mu)\cos(\beta + \mu)}{\cos^2\beta\cos^2 2\mu} = \frac{c^2 - 1}{c^2\cos^2\beta},$$

or

$$\frac{c^2\sin 2\beta}{\sin 2\mu} - \frac{c^2\sin 2\mu\,[\sin 2\mu - \sin 2\beta]}{1 - \sin^2 2\mu} = c^2 - 1,$$

or

$$\sin^3 2\mu + (c^2 - 1)\sin 2\mu - c^2\sin 2\beta = 0.$$

That this cubic will always give at least one real value for μ, is evident on making in the left-hand member $\sin 2\mu$ successively equal to -1, 0, and $+1$; the results obtained are

$-c^2(1 + \sin 2\beta)$, always negative;
$-c^2\sin 2\beta$, negative or positive, according to the sign of $\sin 2\beta$;
$+c^2(1 - \sin 2\beta)$, always positive.

Moreover, it is plain that there is one real value of μ, which makes $\sin 2\mu$ and $\sin 2\beta$ have like signs; this value we shall adopt.

Making, according as c^2 is greater or less than unity,

$$c^2 = \sec^2.\,\gamma, \text{ or } c^2 = \cos^2\gamma',$$

the above cubic is solved by these formulæ (see Chauveuet's Trigonometry, p. 96), it being necessary to make three different cases.

Case I.

$$\tan\phi = \frac{2\sin^2\gamma\tan\gamma}{\sqrt{27}\sin 2\beta},$$

$$\tan\psi = \tan^{\frac{1}{3}}\frac{\phi}{2},$$

$$\sin 2\mu = \frac{2}{\sqrt{3}}\tan\gamma\cot 2\psi.$$

Case II.

$$\sin\phi = \frac{2\sin\gamma'\tan^2\gamma'}{\sqrt{27}\sin 2\beta},$$

$$\tan\psi = \tan^{\frac{1}{3}}\frac{\phi}{2},$$

$$\sin 2\mu = \frac{2}{\sqrt{3}}\sin\gamma'\operatorname{cosec} 2\psi.$$

Case III.

$$\sin 3\phi = \frac{\sqrt{27}\,\sin 2\beta}{2\sin\gamma'\tan^2\gamma'},$$

$$\sin 2\mu = \frac{2}{\sqrt{3}}\sin\gamma'\sin(\phi \pm 60°).$$

When ϕ is impossible in Case II., the formulæ of Case III. must be used; and the upper or lower member of the double sign in the second equation must be taken according as $\sin 2\beta$ is positive or negative; in order that $\sin 2\mu$ may have the same sign with $\sin 2\beta$. All the auxiliary angles ϕ, ψ, and μ may be taken between the limits $\pm 90°$. Since $\sin 2\beta \sin 2\mu$ is always positive, $\tan\beta\tan\mu$ and $\tan\beta\cot\mu$ are so likewise, since they are respectively equivalent to

$$\frac{\sin 2\beta\sin 2\mu}{4\cos^2\beta\cos^2\mu} \text{ and } \frac{\sin 2\beta\sin 2\mu}{4\cos^2\beta\sin^2\mu}.$$

Let us take two auxiliary angles θ and θ', determined by the equations

$$\sin 2\theta = -\frac{\tan^{\frac{1}{2}}\beta\tan^{\frac{1}{2}}\mu\cos\beta\cos 2\mu}{\sin\mu\cos(\beta+\mu)},$$

$$\sin 2\theta' = -\frac{\tan^{\frac{1}{2}}\beta\cot^{\frac{1}{2}}\mu\cos\beta\cos 2\mu}{\cos\mu\sin(\beta-\mu)},$$

or by the equations

$$\sin 2\theta = \mp\frac{\cos 2\mu}{\cos(\beta+\mu)}\sqrt{\frac{\sin 2\beta}{\sin 2\mu}},$$

$$\sin 2\theta' = \mp\frac{\cos 2\mu}{\sin(\beta-\mu)}\sqrt{\frac{\sin 2\beta}{\sin 2\mu}},$$

where the upper or lower of the signs must be taken according as $\frac{\cos\beta}{\sin\mu}$ in the first and $\frac{\cos\beta}{\cos\mu}$ in the second are positive or negative; and 2θ and $2\theta'$ may also be taken within the limits $\pm 90°$. The four values of χ or $\tan\sigma$ are then

$$\begin{aligned}
\tan\sigma &= \tan^{\frac{1}{2}}\beta\tan^{\frac{1}{2}}\mu\tan\theta,\\
\tan\sigma &= \tan^{\frac{1}{2}}\beta\tan^{\frac{1}{2}}\mu\cot\theta,\\
\tan\sigma &= \tan^{\frac{1}{2}}\beta\cot^{\frac{1}{2}}\mu\tan\theta',\\
\tan\sigma &= \tan^{\frac{1}{2}}\beta\cot^{\frac{1}{2}}\mu\cot\theta'.
\end{aligned}$$

If the value of $\sin 2\theta$ or of $\sin 2\theta'$ does not fall within the limits

$\pm$ 1, it indicates that the two corresponding values of $\tan\sigma$ are imaginary. The ambiguity in the determination of σ from its tangent is to be removed by taking it in that quadrant which permits the equation

$$\sin 2\,\sigma = 2\,c\,\sin(\sigma + \beta)$$

to be satisfied.

Although all these roots will satisfy the equations with which we began this discussion, yet they do not all necessarily belong to the problem. The reason of this is, that the three equations are not a complete statement of all the conditions of the problem. If we denote by Δ the distance of the body, whose orbit we are determining, from the earth, we shall have

$$\begin{aligned}
\Delta_{-1} &= r\cos(\chi_0 - \eta - \lambda_{-1}) + R_{-1}\cos(\lambda_{-1} - L_{-1}),\\
\Delta_0 &= r\cos(\chi_0 - \lambda_0) \qquad\quad\; + R_0\cos(\lambda_0 - L_0),\\
\Delta_1 &= r\cos(\chi_0 + \eta - \lambda_1) \;\;\, + R_1\cos(\lambda_1 - L_1).
\end{aligned}$$

The conditions of the problem demand that Δ_{-1}, Δ_0 and Δ_1 shall be essentially positive. Hence, if any system of values of r, χ_0 and η renders any of these quantities negative, it must be rejected. These rejected solutions really belong to the problem when one or more of the quantities λ_{-1}, λ_0 and λ_1 are increased by 180°. In fact, on referring to the equations with which we started, we see they are not altered when any one of the quantities λ is increased by 180°. The geometrical statement of the problem is more comprehensive than the application of it to the discovery of the elements of circular orbits. Instead of the above criteria for the rejection of solutions not applicable, the following, which is simpler, may be used, viz. that χ always must lie in the angle between $L + 180°$ and λ which is less than 180°.

This example is added for the sake of illustration: —

Suppose in the case of Venus revolving about the sun we have these data,

Wash. Mean Time.	λ	L	$\log R$
1869 Jan. 1.0	250° 22′ 59″.1	281° 24′ 54″.9	9.9926528
" June 15.0	94 37 54. 9	84 33 34. 1	0.0069342
" Nov. 27.0	292 3 21. 2	245 32 49. 3	9.9939666

There will be found

$$\begin{aligned}
&\log a_{-1} = 9.7048977_{\mathrm{n}}, \quad \log a_0 = 9.2497072, \quad \log a_1 = 9.8545925,\\
&\log k = 0.5426896\,, \quad \zeta = 324°\,41'\,4.''52, \quad \delta = 176°\,35'\,15.''25,\\
&\log \mathrm{a} = 9.7678074_{\mathrm{n}}, \quad \log \mathrm{b} = 9.3111404.
\end{aligned}$$

Constructing the equilateral hyperbola whose equation is $x y = -1$, and the circle whose radius is 2.89, and the co-ordinates of its centre $x' = +1.69$, $y' = -0.59$, we find the two roots of the equation in σ, $\sigma = 7\frac{1}{2}^\circ$, $\sigma = 241\frac{1}{2}^\circ$. In fact, the value of $a^{\frac{2}{3}} + b^{\frac{2}{3}} = 1.0475$ shows that the equation has, in this case, but two real roots. Pursuing the calculation,

$$\log c = 9.7928205, \quad \beta = 160^\circ\ 44'\ 24''.60, \quad \gamma' = 51^\circ\ 38'\ 20''.85.$$

Case II. is to be used here.

$\phi = -50^\circ\ 40'\ 40''.00$, $\psi = -37^\circ\ 56'\ 3''.23$, $\mu = -34^\circ\ 30'\ 27''.50$ $\theta = 14^\circ\ 49'\ 46''.36$, θ' is impossible, which confirms the preceding statement about the number of real roots; and the values of σ are

$$\sigma = 7^\circ\ 23'\ 36''.95 \text{ and } \sigma = 241^\circ\ 37'\ 18''.04.$$

If we employ the tentative process with the equation

$$\sin 2\sigma = 2c \sin(\sigma + \beta),$$

we shall get $\sigma = 7^\circ\ 23'\ 36''.97$ and $\sigma = 241^\circ\ 37'\ 17''.95$; as these values are more accurate, we shall use them. The two solutions are

$$\begin{aligned} \omega &= 1^\circ\ 16'\ 6''.99, & \omega &= 197^\circ\ 31'\ 54''.15, \\ \log r &= 0.6767422, & \log r &= 9.8624217, \\ \chi_0 &= 272^\circ\ 29'\ 17''.14, & \chi_0 &= 108^\circ\ 45'\ 4''.30, \\ \eta &= 28\ \ 13\ \ 48\ .02, & \eta &= 262\ \ 27\ \ 29\ .00. \end{aligned}$$

On applying the above-mentioned criteria, the first solution is seen to be inadmissible, it makes Δ_0 and Δ_1 negative. If both λ_0 and λ_1 are increased by 180°, the solution will apply. The given example has then but one solution. Below we give a comparison between the values of the elements of Venus's orbit as found in this example, and those of the "Tables"; the differences are of course to be attributed to the neglect of the eccentricity and inclination of the orbit, and in a smaller degree to aberration and perturbations.

	From the Example.	From the Tables.
Mean Distance from the sun	0.7284868	0.7233323
Mean Longitude Jan. 1.0 1869	206° 17′ 35″.30	204° 57′ 28″.89
Mean Motion in Julian Year	2091552″.2	2106641″.438

New Method for facilitating the Conversion of Longitudes and Latitudes of Heavenly Bodies, near the Ecliptic, into Right Ascensions and Declinations, and vice versa.

In the computation of a Lunar Ephemeris, the conversion of the longitudes and latitudes into right ascensions and declinations forms no inconsiderable part of the work to be done. Prof. Hansen, at the end of his "Tables de la Lune," has given some tables, with the view of diminishing the amount of labor required in this conversion.

But their employment seems to me to possess little, if any, advantage over the use of the ordinary formulæ of spherical trigonometry. I propose the following method, which perhaps in a slight degree is more ready than that of Prof. Hansen.

Designating the right ascension, declination, longitude, latitude, and the obliquity of the ecliptic respectively by α, δ, l, b and ϵ, we have the following equations

$$\sin\delta = \cos\epsilon\sin b + \sin\epsilon\cos b\sin l$$

$$= \cos\epsilon\sin b + \frac{\sin\epsilon}{2}\sin(l+b) + \frac{\sin\epsilon}{2}\sin(l-b),$$

$$\tan\frac{\alpha+6^{\text{h}}}{2} = \frac{\cos\frac{\epsilon+b+\delta}{2}}{\cos\frac{\epsilon-(b+\delta)}{2}}\tan\frac{l+90^\circ}{2}.$$

The first equation is well known, the second is easily derived from the known formula, expressed in the usual notation,

$$\tan\frac{A}{2}\tan\frac{B}{2} = \frac{\sin(s-c)}{\sin s},$$

when we remember that, in considering the triangle, formed by the heavenly body and the poles of the equator and ecliptic, A, B, s and c are replaced by $90^\circ+\alpha$, $90^\circ-l$, $90^\circ+\frac{\epsilon-(b+\delta)}{2}$ and ϵ.

Suppose we were to tabulate the functions $\cos\epsilon\sin A$ and $\frac{\sin\epsilon}{2}\sin A$ for a certain value of ϵ (as 23° 27′ 20″ which is nearly its value at present), and in small side tables put the variations of these functions for increments of 1″, 2″. .. .. 9″ in ϵ; we should have the value of $\sin\delta$ by entering the first table with the argument $A = b$, and the second successively with the arguments $A = l + b$ and $A = l - b$, and adding the results thus obtained, after having corrected them for the deviation of the value of ϵ from that adopted in the tables. After which the value of δ could be obtained from a table of natural sines. For the case of the moon, the first function would need tabulation only between the limits $A = 0^\circ$ and $A = 5^\circ\ 17'$; it might be tabulated for every $10''$. The second would have to be tabulated from 0° to 90°; it might be given for every minute of arc. The numbers in these tables might be rendered always positive by adding a constant to them; as, for instance, 0.1 to the first function, and 0.2 to the second; and thus the addition of the three terms of $\sin \delta$ be made easier.

We should then have to subtract 0.5 from the sum, in order to get $\sin \delta$; or we might prepare a special table, which, with the argument $0.5 + \sin \delta$, should give δ. But by the addition of these constants, the extent of the tables would be doubled, as it would be necessary to tabulate the numbers which correspond to negative values of the arguments.

The factor by which $\tan \frac{l + 90^\circ}{2}$ must be multiplied to obtain $\tan \frac{a + 6^h}{2}$ is always positive, and, ϵ being regarded as constant, is a function of $b + \delta$, and, for negative values of $b + \delta$, its value is the reciprocal of that which corresponds to positive values of $b + \delta$. Moreover, when $b + \delta$ is a tolerably small angle, it does not differ much from unity, and varies very uniformly. In the case of the moon $b + \delta$ rarely exceeds the limits $\pm\ 34^\circ$, and the common logarithm of this quantity lies between 9.9447979 and 0.0552021; and its rate of change per minute of arc in $b + \delta$ varies only from 262 to 289 units of the seventh decimal place. We may, with the better advantage, tabulate the function

$$\log \cos \frac{\epsilon - A}{2} - \log \cos \frac{\epsilon + A}{2},$$

for every minute of arc of the argument A from 0° to 34°, with the precept that it is to be subtracted from $\log \tan \frac{l + 90^\circ}{2}$ when $b + \delta$ is a positive angle, but added when $b + \delta$ is negative. It will be necessary to append to the table the variation of the function for a change in ϵ. The functions $\log \tan (45^\circ + \frac{l}{2})$ and $\log \tan (45^\circ + \frac{a}{2})$ can be found from the logarithmic tables, but some labor would be spared had we tables which gave $\log \tan (45^\circ + \frac{A}{2})$ with the argument A both in arc and time; which tables would be useful in many other cases, since this function is frequently met with in trigonometric formulæ.

The modifications necessary in applying this method to the inverse problem of determining the longitude and latitude from the right ascension and declination are obvious. The variations due to the change of the obliquity might perhaps be neglected in using the tables, especially in the case of the declination, and computed at the end by means of the very simple formulæ

$$\frac{d\alpha}{d\epsilon} = -\tan\delta\cos\alpha,$$

$$\frac{d\delta}{d\epsilon} = \sin\alpha.$$

Take this example for illustration: —

On January 14.0, 1871, G. M. T. we have in the case of the moon,

$l = 206^\circ\ 40'\ 35.''9$	$\epsilon = 23^\circ\ 27'\ 19.''81$	
$b = +5\ \ 3\ \ 16.0$	From Tab. I., Arg. b,	$+0.0808224$
	$-1.7 \times (\Delta\epsilon = ^{-0.19})$	0
$l + b = 211\ \ 43\ \ 51.9$	From Tab. II., Arg. $l + b$,	-0.1046706
	$-11.7 \times \Delta\epsilon$	$+2$
$l - b = 201\ \ 37\ \ 19.9$	From Tab. II., Arg. $l - b$,	-0.0733354
	$-8.2 \times \Delta\epsilon$	$+2$
$\delta = -5\ \ 34\ \ 37.16$	$\sin\delta$	-0.0971832
$b + \delta = -0\ \ 31\ \ 21.16$	log tan $148^\circ\ 20'\ 17.''95$	9.7900662_n
$\alpha = 13^h\ 46^m\ 19^s.12$	From Tab. III., to be added,	0.0008223
	$+0.09 \times \Delta\epsilon$	0
	log tan $9^h\ 53^m\ 9^s.56$	9.7908885_n

The objection to this method is, that so many arguments $l + b$, $l - b$, $b + \delta$, $45^\circ + \frac{l}{2}$, and α from $45^\circ + \frac{\alpha}{2}$ are to be formed; but this is confessedly less fatiguing than the taking of tabular quantities from a table.

It may be allowed to notice here a series, which determines α in terms of l, viz.: —

$$\alpha = l + \frac{2}{1}\tan\frac{\epsilon}{2}\tan\frac{b+\delta}{2}\cos l$$

$$- \frac{2}{2}\tan^2\frac{\epsilon}{2}\tan^2\frac{b+\delta}{2}\sin 2\,l$$

$$- \frac{2}{3}\tan^3\frac{\epsilon}{2}\tan^3\frac{b+\delta}{2}\cos 3\,l$$

$$+ \frac{2}{4} \tan^4 \frac{\epsilon}{2} \tan^4 \frac{b + \delta}{2} \sin 4\, l$$

$$+ \&c. \ldots.$$

As $\tan \frac{\epsilon}{2} \tan \frac{b + \delta}{2}$, in the case of the moon, is always between the limits $\pm \frac{2}{31}$, the above series is, for this body, quite convergent.

I add the values of the function $\log \dfrac{\cos \frac{\epsilon - A}{2}}{\cos \frac{\epsilon + A}{2}}$, computed for every degree from 0° to 35° of the argument A and for $\epsilon = 23° \; 27' \; 20''$.

A	$\log \dfrac{\cos \frac{\epsilon - A}{2}}{\cos \frac{\epsilon + A}{2}}$	Δ	Δ^2	Change of this function for an inc. in ϵ of $1''$ in units of the seventh decimal.
0°	.0000000	+ 15736		+ 0.00
1	.0015736	15738	+ 2	0.19
2	.0031474	15744	6	0.38
3	.0047218	15751	7	0.57
4	.0062969	15761	10	0.77
5	.0078730	15773	12	0.96
6	.0094503	15789	16	1.15
7	.0110292	15806	17	1.34
8	.0126098	15826	20	1.54
9	.0141924	15849	23	1.73
10	.0157773	15874	25	1.92
11	.0173647	15902	28	2.12
12	.0189549	15933	31	2.31
13	.0205482	15965	32	2.50
14	.0221447	16002	37	2.70
15	.0237449	16040	38	2.89
16	.0253489	16081	41	3.09
17	.0269570	16124	43	3.28
18	.0285694	16172	48	3.48
19	.0301866	16221	49	3.68
20	.0318087	16273	52	3.88
21	.0334360	16328	55	4.08
22	.0350688	16386	58	4.28
23	.0367074	16447	61	4.48
24	.0383521	16511	64	4.68
25	.0400032	16578	67	4.88
26	.0416610	16648	70	5.08
27	.0433258	16721	73	5.29
28	.0449979	16797	76	5.49
29	.0466776	16877	80	5.70
30	.0483653	16959	82	5.90
31	.0500612	17046	87	6.11
32	.0517658	17135	89	6.32
33	.0534793	17228	93	6.53
34	.0552021	+ 17325	+ 97	6.74
35	0.0569346			+ 6.95